Part A

1. (10 points) If a sequence below converges, find its limit, and justify by citing any theorems/rules you use. If a sequence below diverges, state whether it diverges because it oscillates, diverges to $+\infty$, or diverges to $-\infty$.

(a) $a_n = \left(\frac{\cos(n)}{n}\right)^2$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\cos(n)}{n}\right)^2 = \lim_{n\to\infty}\frac{\cos^2(n)}{n^2} \overset{\uparrow}{=} \boxed{0 \qquad \text{CONV}}$$

Since $\lim_{n\to\infty}\frac{1}{n^2}=0$

$0 \le \cos^2(n) \le 1$

$0 \le \frac{\cos^2(n)}{n^2} \le \frac{1}{n^2} \Rightarrow 0 \le \lim_{n\to\infty}\frac{\cos^2(n)}{n^2} \le \lim_{n\to\infty}\frac{1}{n^2}$ $\boxed{\text{by sq. thm.}}$

(b) $a_n = (-e)^n$ $= (-1)^n e^n$

$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} e^n = \infty$ DIV, so ~~sq~~. thm. doesn't apply.

Since a_n alternates +/[illegible], $\lim_{n\to\infty} a_n$ $\boxed{\text{DIV b/c OSCL.}}$

(c) $a_n = \frac{-2e^n+\sqrt{n}}{e^n+1}$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty}\frac{-2e^x+\sqrt{x}}{e^x+1} = \lim_{x\to\infty}\frac{-2+\overset{\to 0 \text{ as } x\to\infty \text{ (see below)}}{\sqrt{x}/e^x}}{1+\underset{\to 0 \text{ as } x\to\infty}{\cancel{1/e^x}}} = \frac{-2}{1} = \boxed{-2}$$

$\boxed{\text{CONV}}$

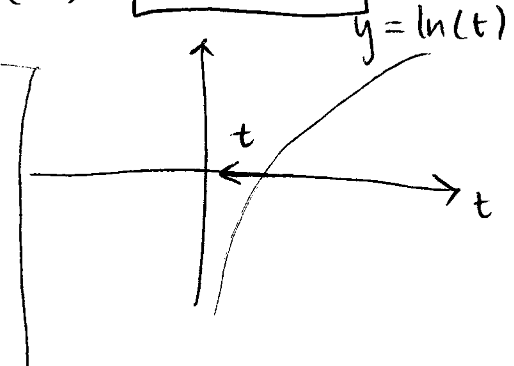
$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^x} \underset{\frac{\infty}{\infty}\ \text{l'HR}}{=} \lim \frac{\frac{1}{2}x^{-1/2}}{e^x} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x} = 0$$

(d) $a_n = \ln\left(\frac{n}{n^2+1}\right)$ $\Rightarrow \lim_{n\to\infty} a_n = \lim_{t\to 0^+}\ln(t) = \boxed{-\infty \text{ DIV}}$

as $n \to \infty$, $\frac{n}{n^2+1} \to 0^+$

by l'HR or dividing top & bottom

by n...



2. (10 points) Determine whether the following series converge absolutely, converge only conditionally, or diverge, naming any tests you use, and justifying their use completely.

(a)

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$$

$f(x) = \frac{1}{x\ln(x)^2}$ is pos., decr., cont's for $x \geq 2$.

integral test: $\int_2^\infty \frac{1}{x\ln(x)^2}dx = \lim_{b\to\infty}\int_2^b \frac{1}{x\ln(x)^2}dx$

$$\underset{\uparrow}{=} \lim_{b\to\infty}\int_{\ln(2)}^{\ln(b)} \frac{1}{u^2}\,du = \lim_{b\to\infty}\left[-u^{-1}\right]_{\ln(2)}^{\ln(b)}$$

$u=\ln(x)$
$du=\frac{1}{x}dx$

$$= \lim_{b\to\infty}\left[-\overset{\nearrow 0}{\frac{1}{\ln(b)}} + \frac{1}{\ln(2)}\right] = \frac{1}{\ln(2)} \text{ CONV}$$

So, by integral test, $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)^2}$ $\boxed{\text{conv (abs)}}$ as well.
$\uparrow$
because all terms are positive anyway.

(b)

$$\sum_{n=1}^{\infty} (\ln(4n^2+3n+2) - \ln(4n^2+5n+6))^n$$

root test:

$$\lim_{n\to\infty}|a_n|^{\frac{1}{n}} = \lim_{n\to\infty}\left|\ln\left(\frac{4n^2+3n+2}{4n^2+5n+6}\right)\right| \underset{\uparrow}{=} \left|\ln\left(\lim_{n\to\infty}\left(\frac{4n^2+3n+2}{4n^2+5n+6}\right)\right)\right|$$

continuous function theorem

$$= \left|\ln\left(\lim_{n\to\infty}\left(\frac{4+\overset{0}{\cancel{3/n}}+\overset{0}{\cancel{2/n^2}}}{4+\underset{0}{\cancel{5/n}}+\underset{0}{\cancel{6/n^2}}}\right)\right)\right| = |\ln(1)| = \boxed{0} < 1$$

So the series $\boxed{\text{conv. abs.}}$ by root test

3. (10 points) Determine whether the following series converge absolutely, converge only conditionally, or diverge, naming any tests you use, and justifying their use completely.

(a)

$$\sum_{n=1}^{\infty} \frac{n4^n + \sqrt{n}}{3^n - 7} = \sum a_n$$

① $b_n = \frac{4^n}{3^n} \Rightarrow \sum b_n$ DIV by GST with $r = \frac{4}{3} > 1$

② $a_n \geq \frac{n4^n}{3^n} \geq \frac{4^n}{3^n} = b_n$ so

↑

LHS has larger num. and smaller denom.

③ $\sum a_n$ $\boxed{\text{DIV}}$ as well by CT.

(b)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(3^n)} = \sum a_n$$

test for abs conv first:

Ⓐ $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\ln(3^n)} = \sum \frac{1}{\ln(3) \cdot n} = \frac{1}{\ln(3)} \sum \frac{1}{n}$ DIV by p-test w/ $p=1$

$\boxed{(\ln(3^n) = n \cdot \ln(3))}$

and (nonzero) const. mult. of DIV is DIV

so $\sum a_n$ is $\boxed{\text{NOT abs. conv.}}$

Ⓑ ① $a_n = (-1)^n \frac{1}{\ln(3^n)}$ and $\frac{1}{\ln(3^n)} \geq 0$ so $\sum a_n$ is alternating ✓

② $f(x) = \frac{1}{\ln(3^x)} = \frac{1}{x\ln(3)} \Rightarrow f'(x) = -\frac{1}{x^2} \cdot \frac{1}{\ln(3)} < 0$ for all $x \Rightarrow$

$|a_n|$ is decreasing ✓

③ $\lim_{n\to\infty} \frac{1}{\ln(3^n)} = \lim_{n\to\infty} \frac{1}{n\ln(3)} = \frac{1}{\ln(3)} \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

So $\sum a_n$ $\boxed{\text{conv.}}$ by AST.

Ⓐ & B $\Rightarrow$ $\sum a_n$ is $\boxed{\text{COND CONV}}$

4. **(10 points)** Find the radius and interval of convergence of the power series below.

(a) $\sum_{n=1}^{\infty} \frac{3^n(2n^2+1)(x-3)^n}{2^n(2n)!}$.

ratio test: $\lim_{n\to\infty}\left|\frac{3^{n+1}(2(n+1)^2+1)(x-3)^{n+1}}{2^{n+1}\,(2(n+1))!}\cdot\frac{2^n(2n)!}{3^n(2n^2+1)(x-3)^n}\right|$

$$= \lim_{n\to\infty}\left|\frac{3}{2}\cdot\overset{\to 0}{\frac{(x-3)}{(2n+3)(2n+2)}}\cdot\overset{\to 1}{\frac{(2(n+1)^2+1)}{(2n^2+1)}}\right| = 0 < 1 \text{ for all } x$$

So $R=\infty$, IOC $= (-\infty,\infty)$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n(x+3)^n}{4^n\sqrt{n}}$.

ratio test: $\lim_{n\to\infty}\left|\frac{-1}{4}\cdot(x+3)\cdot\overset{\to 1}{\frac{\sqrt{n}}{\sqrt{n+1}}}\right| = \left|\frac{x+3}{4}\right| < 1$

when $-1 < \frac{x+3}{4} < 1$ or $-4 < x+3 < 4$ R

or $-7 < x < 1$

endpts: $x=-7$ $\sum_{n=1}^{\infty}\frac{(-1)^n(-4)^n}{4^n\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DIV by p-test $p=\frac{1}{2}<1$

$x=1$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ CONV by AST alt. ✓ decr. ✓ $|a_n|\to 0$ ✓

$\boxed{R=4}$ $\boxed{\text{IOC} = (-7,1]}$

Check: $\frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}(1-2x)^{-1} = -1(1-2x)^{-2}(-2) = \frac{2}{(1-2x)^2}$ ✓

5. (10 points) Consider the function $f(x) = \dfrac{2}{(1-2x)^2}$.

(a) Write out the first five nonzero terms, and express in sigma notation a power series expansion for $f(x)$ about $x = 0$.

$$f(x) = \frac{d}{dx}\left(\frac{1}{1-2x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty}(2x)^n\right) = \sum_{n=0}^{\infty} 2^n \frac{d}{dx} x^n$$

geometric series formula, true for $|2x|<1$ or $R=\frac{1}{2}$, $IOC=(-\frac{1}{2},\frac{1}{2})$

$(2x)^n = 2^n x^n$ and constants pull out of derivatives

$$= \sum_{n=1}^{\infty} 2^n \cdot n x^{n-1} \Rightarrow f(x) = \sum_{n=1}^{\infty} n 2^n x^{n-1}$$

get rid of $n=0$ term since it's zero.

$$= 2 + 2\cdot2^2x + 3\cdot2^3x^2 + 4\cdot2^4x^3 + 5\cdot2^5x^4 + \cdots$$

$\boxed{R=\frac{1}{2} \text{ by int./diff. thm.}}$

(b) What are the radius and interval of convergence of the series you found in (a)?

endpts. $x = -\frac{1}{2}$ $\quad \sum_{n=1}^{\infty} 2(-1)^{n-1}n$ DIV by test for div.

$x = \frac{1}{2}$ $\quad \sum_{n=1}^{\infty} 2n$ " "

$2^n\left(-\frac{1}{2}\right)^{n-1} =$

$2(-1)^{n-1}\cancel{2}^{n-1}\cancel{\left(\frac{1}{2}\right)}^{n-1} =$

$2(-1)^{n-1}$

$\boxed{IOC = (-\frac{1}{2}, \frac{1}{2})}$

6. (10 points) Consider the function $f(x) = \ln(2x)$.

(a) Write out the first five nonzero terms, and express in sigma notation the Taylor series expansion for $f(x)$ about $x = 3$.

$$f(x) = \ln(2x)$$
$$f'(x) = \frac{1}{2x} \cdot 2 = \frac{1}{x} = x^{-1}$$
$$f''(x) = -\frac{1}{x^2} = -x^{-2}$$
$$f'''(x) = 2x^{-3}$$
$$f^{(4)}(x) = 2 \cdot (-3)\, x^{-4}$$

$$f(3) = \ln(6)$$
$$f'(3) = \frac{1}{3}$$
$$f''(3) = -\frac{1}{3^2}$$
$$f'''(3) = 2 \cdot \frac{1}{3^3}$$
$$f^{(4)}(3) = -2 \cdot 3 \cdot \frac{1}{3^4}$$

$$C_0 = \ln(6)$$
$$C_1 = \frac{1}{3}$$
$$C_2 = -\frac{1}{3^2} \cdot \frac{1}{2!}$$
$$C_3 = 2 \cdot \frac{1}{3^3} \cdot \frac{1}{3!}$$
$$C_4 = -2 \cdot 3 \cdot \frac{1}{3^4} \cdot \frac{1}{4!}$$

$$\underline{n \geq 1}: \; C_n = (-1)^{n-1} \frac{(n-1)!}{3^n \cdot n!} = (-1)^{n-1} \frac{1}{3^n \cdot n}$$

$$\ln(6) + \underline{\tfrac{1}{3}(x-3)} + \underline{\tfrac{-1}{3^2 2}(x-3)^2} + \underline{\tfrac{1}{3^3 \cdot 3}(x-3)^3} + \underline{\tfrac{-1}{3^4 4}(x-3)^4} + \cdots = \ln(6) + \sum_{n=1}^{\infty} \underline{(-1)^{n-1} \tfrac{1}{3^n \cdot n}(x-3)^n}$$

(b) What are the radius and interval of convergence of the series you found in (a)?

$\underline{\text{ratio test}}$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n-1}(x-3)^{n+1}}{3^{n+1}(n+1)} \cdot \frac{3^{n}(n)}{(-1)^n (x-3)} \right| = \lim_{n\to\infty} \left| \frac{(x-3)}{3} \left(\cancel{\frac{n}{n+1}} \right) \right| \xrightarrow{} 1$$

$$= \left| \frac{x-3}{3} \right| < \underline{1} \quad \text{for } -1 < \frac{x-3}{3} < 1 \quad \text{or} \quad -3 < x - 3 < 3$$

$$\boxed{R = 3}$$

$$\Rightarrow 0 < x < 6$$

$\underline{\text{endpts}}$. $\underline{x = 0}$: $-\sum_{n=1}^{\infty} \frac{1}{n}$ DIV harmonic

$\underline{x = 6}$ $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ AST conv.

$$\boxed{IOC = (0, 6]}$$

7. (10 points) Find the sum of the following convergent series. You do not need to justify that they converge.

(a) $8+4+2+1+\frac{1}{2}+\frac{1}{4}+\cdots$ $= \frac{a}{1-r} = \frac{8}{1-\frac{1}{2}} = \frac{8}{\frac{1}{2}} = 2\cdot 8 = \boxed{16}$

geom. $r=\frac{1}{2}$, $a=8$

(b) $\sum_{n=1}^{\infty}\left[\sin\left(\frac{1}{n}\right)-\sin\left(\frac{1}{n+1}\right)\right]$ telescoping.

$$S_k = \sin(1) - \cancel{\sin\left(\tfrac{1}{2}\right)} + \cancel{\sin\left(\tfrac{1}{2}\right)} - \cancel{\sin\left(\tfrac{1}{3}\right)} + \cdots + \cancel{\sin\left(\tfrac{1}{k}\right)} - \sin\left(\tfrac{1}{k+1}\right)$$

$$= \sin(1) - \sin\left(\tfrac{1}{k+1}\right)$$

$$\lim_{k\to\infty}\left(\sin(1) - \sin\left(\tfrac{1}{k+1}\right)\right) = \sin(1) - \sin(0) = \boxed{\sin(1)}$$

(c) $\sum_{n=0}^{\infty}\frac{(-1)^n(4)^{2n}}{3^{2n}(2n)!}$ $= \cos\left(\frac{4}{3}\right)$

(d) $\sum_{n=0}^{\infty}\frac{(-1)^n 5^{2n+1}}{(2n+1)8^{2n+1}}$ $= \arctan\left(\frac{5}{8}\right)$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

8. (10 points) Consider the function $f(x) = \dfrac{3x - \sin(3x)}{x^3}$.

(a) Find the first five nonzero terms of the Taylor series expansion of $f(x)$ about $x = 0$.

$$\frac{3x - \sin(3x)}{x^3} = \frac{\cancel{3x} - \left(\cancel{3x} - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots\right)}{x^3}$$

$$= \frac{1}{x^3}\left(\frac{(3x)^3}{3!} - \frac{(3x)^5}{5!} + \frac{(3x)^7}{7!} - \frac{(3x)^9}{9!} + \cdots\right)$$

$$= \left(\frac{3^3}{3!} - \frac{3^5x^2}{5!} + \frac{3^7x^4}{7!} - \frac{3^9x^6}{9!} + \frac{3^{11}x^8}{11!} - \cdots\right)$$

don't have to do

$$\overset{\downarrow}{=} \sum_{n=1} \frac{(-1)^{n-1}3^{2n+1}x^{2n-2}}{(2n+1)!}$$

(b) What is the value of $f^{(5)}(0)$?

$$f^{(5)}(0) = c_5 \cdot 5! = 0 \cdot 5! = \boxed{0}$$

(c) What is the value of $f^{(6)}(0)$?

$$f^{(6)}(0) = c_6 \cdot 6! = \boxed{-\frac{3^9}{9!}6!}$$

(d) What is the value of $\lim_{x \to 0} f(x)$?

$\boxed{\frac{3^3}{3!}}$ all other terms → 0

(e) What is the Taylor polynomial of degree 4 of $f(x)$ at $x = 0$?

$$\frac{3^3}{3!} - \frac{3^5x^2}{5!} + \frac{3^7x^4}{7!}$$

Part B

9. (10 points) Consider the parametric curve defined by

$$x = t^2$$

$$y = t^3 - t.$$

(a) For which values of t does the curve have a horizontal tangent line?

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-1}{2t} = 0$ when $3t^2-1=0$, $2t\neq 0$

$3t^2=1$ when $t^2=\frac{1}{3}$ or $\boxed{t=\pm\frac{1}{\sqrt{3}}}$

(b) For which values of t does the curve have a vertical tangent line?

$2t=0$ but $3t^2-1\neq 0$ or $\boxed{t=0}$

(c) Find the tangent line at $t = 2$.

$\left.\frac{dy}{dx}\right|_{t=2} = \frac{3\cdot 4-1}{4} = \frac{11}{4}$ = slope

$(x_0,y_0)=(x(2),y(2))=(4,8-2)=(4,6)$

tangent line : $\boxed{y-6=\frac{11}{4}(x-4)}$

(d) Determine intervals of t-values for which the parametric curve is concave up and intervals for which it is concave down.

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{3t^2-1}{2t}\right)}{2t} = \frac{6t(2t)-(3t^2-1)2}{(2t)^3} = \frac{12t^2-6t^2+2}{(2t)^3}$

$= \frac{6t^2+2}{(2t)^3}$ ← always +

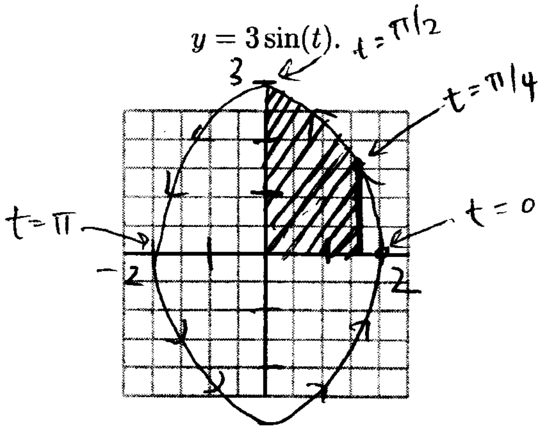
← neg. when $t<0$ ⟶ conc. down for $(-\infty,0)$

pos. when $t>0$ ⟶ conc. up for $(0,\infty)$

10. (10 points) Consider the parametric curve defined by

$$x = 2\cos(t)$$

$$y = 3\sin(t).$$



(a) Sketch this curve on the graph above, indicating the direction of increasing t.

(b) Fill in the area under the curve from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{2}$ on your sketch above.

(c) Find the area under the curve from $t = \frac{\pi}{4}$ to $t = \frac{\pi}{2}$ using an appropriate integral .

$$\int_{\frac{\pi}{4}}^{\pi/2} y\,dx = \int_{\frac{\pi}{4}}^{\pi/2} 3\sin(t)\left(-2\sin(t)\right)dt$$

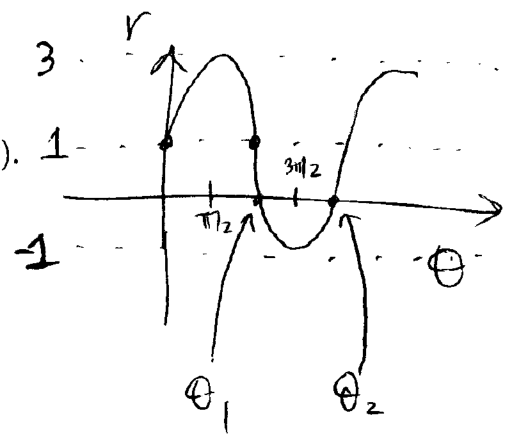
$$= -6\int_{\pi/4}^{\pi/2} \sin^2 t\,dt = -6\int_{\frac{\pi}{4}}^{\pi/2} \frac{1}{2}\left(1-\cos(2t)\right)dt$$

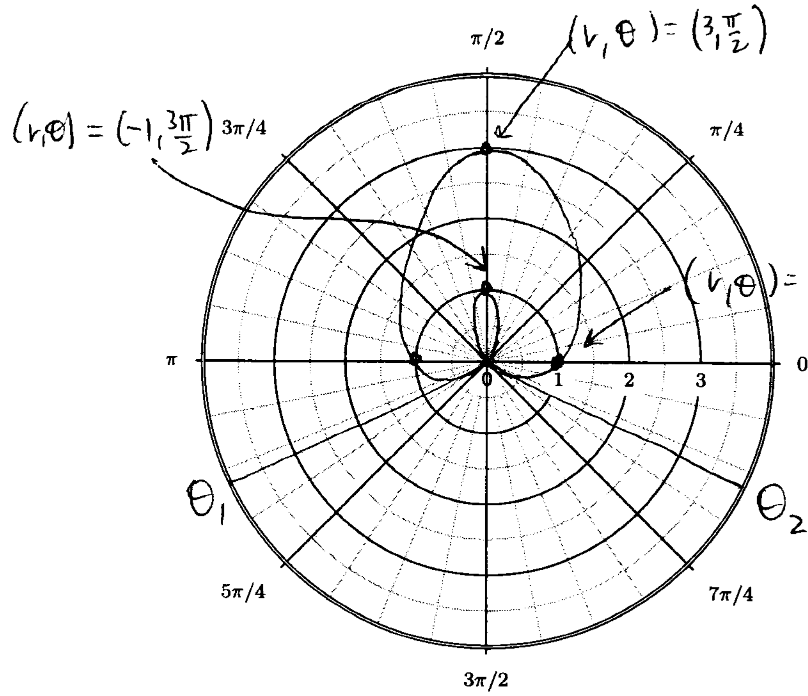
$$= -3\left[t - \frac{1}{2}\sin(2t)\right]_{\frac{\pi}{4}}^{\pi/2} = -3\left[\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right]$$

$$= -3\left(\frac{\pi}{4} + \frac{1}{2}\right)$$ (neg. because integrating R to L on x-axis)

area = $\boxed{3\left(\frac{\pi}{4} + \frac{1}{2}\right)}$

11. (10 points) Consider the polar curve defined by $r = 1 + 2\sin(\theta)$.

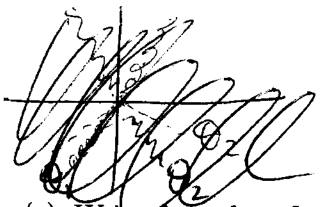


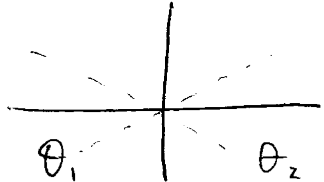


(a) Draw a clear sketch of the curve above.

(b) At which angles does the curve cross itself? when $r = 0$, or $1+2\sin\theta = 0 \Rightarrow$

$\sin\theta = -\frac{1}{2} \Rightarrow \theta_1 = \boxed{\frac{7\pi}{6}}$, $\theta_2 = \boxed{\frac{11\pi}{6}}$

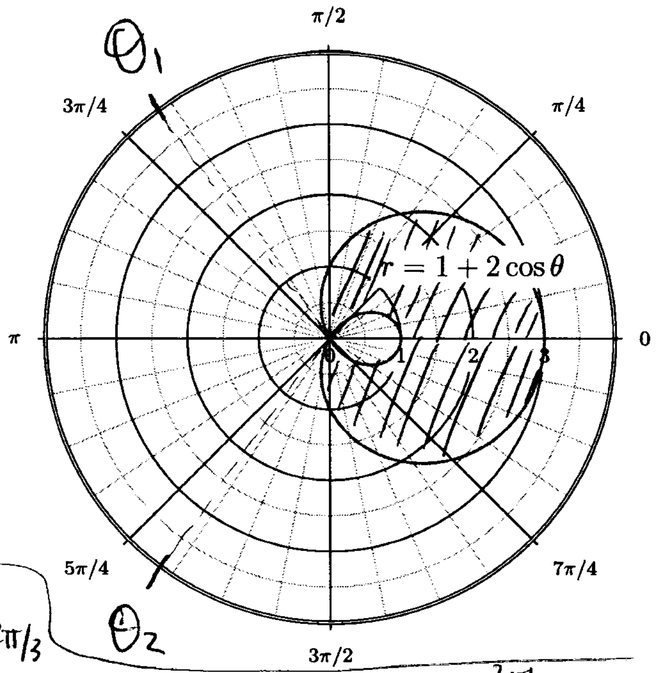




(c) Write down but **do not evaluate** an integral that would give the arc length of the curve from $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1+2\sin\theta)^2}\, d\theta = \int_0^{2\pi} \sqrt{4\cos^2\theta + 1 + 4\sin\theta + 4\sin^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{5 + 4\sin\theta}\, d\theta$$

12. (10 points) Consider the polar curve defined by $r = 1 + 2\cos(\theta)$. Find the area inside the larger loop, but outside the smaller loop of this curve.



r=0 when

$1+2\cos\theta = 0$ or

$\cos\theta = -\frac{1}{2}$

so $\theta_1 = \boxed{\frac{2\pi}{3}}$

$\theta_2 = \boxed{\frac{4\pi}{3}}$

$$A_{outer} = 2\int_0^{\theta_1} \frac{1}{2} r^2 \, d\theta = 2\int_0^{2\pi/3} \frac{1}{2}(1+2\cos\theta)^2 \, d\theta = \int_0^{2\pi/3} (1 + 4\cos\theta + 4\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi/3} [1 + 4\cos\theta + 2(1+\cos(2\theta))]\, d\theta = \int_0^{2\pi/3} (3 + 4\cos\theta + 2\cos 2\theta)\, d\theta$$

$$= \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_0^{2\pi/3} = \cancel{3}(2\pi/\cancel{3}) + 4\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{4\pi}{3}\right) - 0$$

$$= 2\pi + 4\left(\frac{\sqrt{3}}{2}\right) + \left(-\frac{\sqrt{3}}{2}\right) = \boxed{2\pi + \frac{3\sqrt{3}}{2}}$$

$$A_{inner} = \cancel{2}\int_{\theta_1}^{\pi} \cancel{\frac{1}{2}} r^2 \, d\theta = \Big[3\theta + 4\sin\theta + \sin 2\theta\Big]_{\frac{2\pi}{3}}^{\pi} = (3\pi + 0 + 0) - \left(2\pi + \frac{3\sqrt{3}}{2}\right)$$

$$= \boxed{\pi - \frac{3\sqrt{3}}{2}}$$

$$A_{outer} - A_{inner} = 2\left(2\pi + \frac{3\sqrt{3}}{2}\right) - 3\pi = \boxed{\pi + 3\sqrt{3}}$$